A note on the velocity profile and longitudinal mixing in a broad open channel

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The mean velocity profile near the surface of turbulent flow in a broad open channel is discussed with dimensional arguments, and a new empirical constant m is introduced which is analogous to von Kármán's constant for flow near a rigid boundary. It is shown that, while the velocity profile depends only rather weakly on m, the dependence of the coefficient of apparent longitudinal diffusion is stronger, and measurements of diffusion could, in principle, provide an accurate determination of its value. The new profiles for various values of m are compared with those in current use, and finally the correction for finite Reynolds number is discussed.

1. Introduction

In a recent paper Elder (1959) discussed turbulent mixing in a broad open channel on the basis of Taylor's (1954) theory and calculated a coefficient of apparent longitudinal diffusion in agreement with observation on the assumption of a logarithmic mean velocity profile. This form is not theoretically satisfactory near a free surface, and since the coefficient of apparent longitudinal diffusion depends strongly on the shape of the profile, some further investigation seems desirable.

In this note a more satisfactory form for the profile is sought. It is shown that dimensional reasoning, which in this context is equivalent to mixing-length theory, leads to a new formula containing one unknown constant. This constant plays a role at a free surface analogous to that of von Kármán's constant at a fixed wall, and is named 'the mixing constant at a free surface'. Rather precise measurements of the velocity profile would be needed for a direct determination of the value of the constant, but the more marked dependence of the apparent longitudinal coefficient suggests that it may be found more easily from measurements of diffusion.

There are a large number of complications which may limit the usefulness of the present theory in practical situations. It was first developed some years ago in connexion with the diffusion of salt water in an estuary (Hughes 1958), but it turned out to be irrelevant in that case owing to the large effect of the density difference between the fresh and salt water. In artificial channels the velocity profile is commonly distorted by a secondary flow due to their finite width; and in all cases any relative motion between the water surface and the air above it may lead to a spurious surface current or even the generation of waves. Nevertheless, it still seems worth while to study the idealized case in which these complications are absent, for until that is understood, there is little hope of deepening our knowledge of real flows no matter how good empirical formulae may be.

2. The velocity profile near a free surface

In a broad open channel containing a steadily flowing fluid of constant density the shear stress varies linearly with height and vanishes at the free surface. So, if z is the height above the bottom, the friction velocity u_* is given by

$$u_*^2 = u_{*0}^2(1 - z/h), \tag{1}$$

or

$$u_* = q u_{*0}, \tag{2}$$

where $q = (1 - z/h)^{\frac{1}{2}}$. Now if the Froude number $u_{*0}(gh)^{-\frac{1}{2}}$ is so small that the surface remains flat and the Reynolds number $u_{*0}h/\nu$ is so large that molecular viscosity may be neglected, and if it is assumed that the components of the turbulence which contribute to the eddy viscosity near the surface are determined entirely by local quantities without any direct dependence on the total depth of the flow, the usual dimensional arguments assert that the eddy viscosity is given by

$$\begin{aligned} K &= m u_*(h - z) \\ &= m h u_{*0} q^3, \end{aligned} \tag{3}$$

where m is the unknown constant which has already been referred to as 'the mixing constant at a free surface'; the corresponding quantity near a rigid wall is von Kármán's constant k. It follows from (3) that the velocity near the surface must be given by

$$u = u_{\max} - 2m^{-1}u_{*0}(1 - z/h)^{\frac{1}{2}}. \tag{4}$$

The logarithmic profile assumed by Elder does not have this form, and seems to lack justification.

There is no difficulty in obtaining a crude estimate of the correction that must be applied to (4) when the Reynolds number is not so large that it can be neglected, and this will be done in § 5.

The flow near the free surface differs from that in the constant stress region near the floor in two distinct ways, each of which may cause m to differ from k. First, the stress at the surface is zero so that there is a gradient of turbulent energy and therefore the 'diffusion' and 'pressure flow' terms in the turbulent energy balance may be important. This situation also occurs at a point of separation of a turbulent boundary layer as has been discussed by Stratford† (1959*a*, *b*) who gives a formula equivalent to (3). He has also been able to produce an experimental flow with zero wall stress over a finite distance but which is necessarily complicated by the deceleration produced by a pressure gradient. From his determination of the variation of mixing length *very* near the wall, it would seem

† I am indebted to a referee for calling my attention to this.

that m is greater than k, but it is not possible to obtain accurate numerical values from his paper.†

Secondly, a free surface differs from a rigid wall because tangential movements are not prohibited at the boundary, although it is doubtful whether such movements affect the eddy viscosity appreciably. At first sight it might be thought that all turbulent velocity at the surface was excluded by dimensional reasoning of the same form as that used for the eddy viscosity, but this is not so if the movements extend in scale up to sizes comparable with the depth of the channel, since these scales are not covered by the dimensional argument. The point is quite a subtle one and outside the main theme of this paper; a brief explanation is attempted in the appendix.

3. The velocity profile in an open channel

We know that near the floor of the channel

$$K = ku_{*0}z; \tag{5}$$

so, in order to obtain an approximation to K throughout the depth of the channel, we may conveniently fit a polynomial in q such that (3) is satisfied at the surface and (5) at the floor. Thus

$$\frac{K}{hu_{*0}k} = \frac{m}{k}q^3(1-q^2)\left(1-\frac{m-k}{m}q^2\right)$$
$$= \frac{q^3(1-q^2)(b^2-q^2)}{(b^2-1)}, \tag{6P}$$

where $$b = \left(\frac{m}{m-k}\right)^{\frac{1}{2}}.$$

This formula may be compared with that resulting from the logarithmic profile

$$\frac{K}{hu_{*0}k} = q^2(1-q^2); \tag{6L}$$

and with that resulting from an application of von Kármán's hypothesis to this flow (see, for example, Hunt 1954):

$$\frac{K}{hu_{*0}k} = 2q^2(1-q). \tag{6K}$$

The velocity profiles corresponding to (6P), (6L) and (6K) may readily be found. They are

$$\frac{k(u-U)}{u_{*0}} = \frac{b^2-1}{b}\ln\frac{b+1}{b-1} - \ln\frac{1+q}{1-q} + b^{-1}\ln\frac{b+q}{b-q}, \tag{7P}$$

$$\frac{k(u-U)}{u_{*0}} = 1+\ln(1-q^2), \tag{7L}$$

$$\frac{k(u-U)}{u_{*0}} = \tfrac{5}{6}+q+\ln(1-q), \tag{7K}$$

† The fact that in contrast to this the quantity β used by Stratford to represent (an average of) m/k in his inner layer is less than unity possibly suggests that in his case the range of validity of (3) is less than his theory implies.

where U is the mean (discharge) velocity. U itself is to be found from the condition that $u = 0$ when $z = z_0$; so, since z_0 is very much smaller than h,

$$\frac{kU}{u_{*0}} = \ln\frac{h}{z_0} + \ln 4 - b\ln\frac{b+1}{b-1}, \tag{8P}$$

$$\frac{kU}{u_{*0}} = \ln\frac{h}{z_0} - 1, \tag{8L}$$

$$\frac{kU}{u_{*0}} = \ln\frac{h}{z_0} - \tfrac{11}{6} + \ln 2$$

$$= \ln\frac{h}{z_0} - 1{\cdot}140. \tag{8K}$$

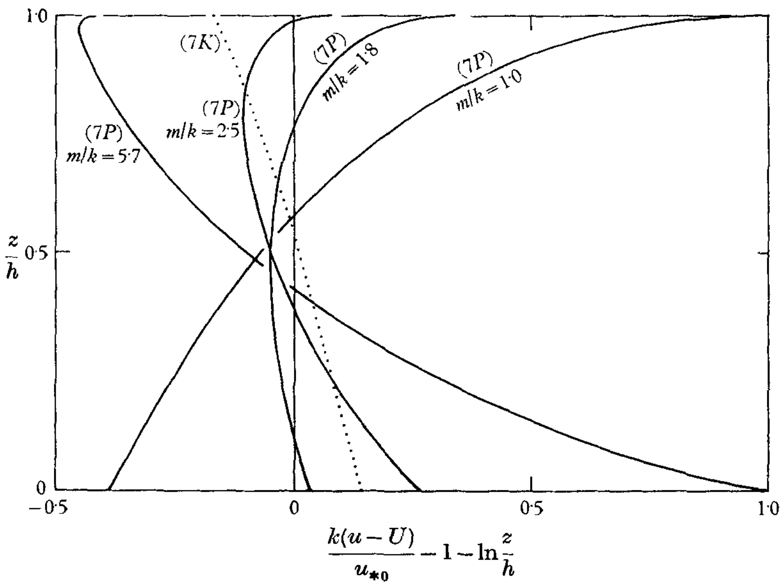

FIGURE 1. Difference between velocity profiles for various values of the mixing constant at the free surface and the logarithmic form. The dotted curve represents the von Kármán profile in the same way.

The fact that (8L) and (8K) are commonly found satisfactory for engineering purposes yields a very crude method of estimating b. If we set

$$\ln 4 - b\ln\frac{b+1}{b-1}$$

equal to $-1{\cdot}07$, so that (8P) coincides with the mean of (8L) and (8K), we find $b = 1{\cdot}45$ and so $m/k = 1{\cdot}91$.

The velocity profiles (7P), (7L) and (7K) are so alike that there is little prospect of deciding between them on the basis of direct measurements of velocity. In order to bring out the small differences most clearly, the difference between (7L) and (7K) and that between (7L) and (7P) for various values of m/k are plotted in figure 1.

4. The apparent longitudinal diffusion coefficient

For a two-dimensional flow it can easily be shown (for example, by integrating equation (9) of Elder's paper by parts) that the coefficient of apparent longitudinal diffusion is given by

$$D = \int_0^1 K^{-1}\left[\int_z^1 (u-U)\,dz\right]^2 dz. \tag{9}$$

From (7P), (7L) and (7K) one finds respectively

$$\int_z^1 (u-U)\,dz = q^2\frac{b^2-1}{b}\ln\frac{b+1}{b-1}+(1-q^2)\ln\frac{1+q}{1-q}-\frac{b^2-q^2}{b}\ln\frac{b+q}{b-q}, \tag{10P}$$

$$\int_z^1 (u-U)\,dz = -(1-q^2)\ln(1-q^2), \tag{10L}$$

$$\int_z^1 (u-U)\,dz = \tfrac{2}{3}q^3+\tfrac{1}{3}q^2-q-(1-q^2)\ln(1-q). \tag{10K}$$

So (9) can readily be calculated numerically in each case. The results are shown in figure 2. Elder's measurements gave a value of $6{\cdot}06u_{*0}h$ for D, which with $k = 0{\cdot}40$ would correspond to $m = 0{\cdot}80$. However, in the present writer's opinion it is unlikely that sufficiently accurate measurements could be taken with the apparatus at Elder's disposal and the resulting value of m should be viewed with caution. It does, nevertheless, confirm the expectation that m is greater than k.

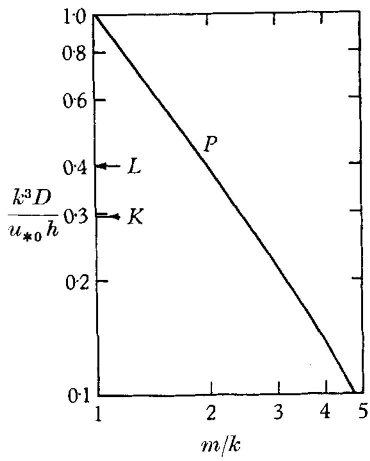

FIGURE 2. The apparent longitudinal diffusion coefficient as a function of the mixing constant at the free surface.

5. The correction for finite Reynolds number

The peaks on the velocity profile at the surface in figure 1 are due to the way in which K falls to zero; and it is important to inquire how they are modified by molecular viscosity, since in reality it is not possible for K to fall below ν. Fortunately the correction which has to be applied is small in most practical cases and can be estimated. The following conventional analysis is certainly not exactly correct but should be adequate for most cases.

Let us write the total effective viscosity K as the sum of a turbulent part K_T and a molecular part ν; and also express the shear stress u_*^2 as the sum of a turbulent part u_{*T}^2 and a molecular part u_{*m}^2. Thus

$$u_*^2 = u_{*T}^2 + u_{*m}^2 = K_T \frac{du}{dz} + \nu \frac{du}{dz} = q^2 u_{*0}^2. \tag{11}$$

If we still use (3) for the turbulent part,

$$K_T = mhq^2 u_{*T}; \tag{12}$$

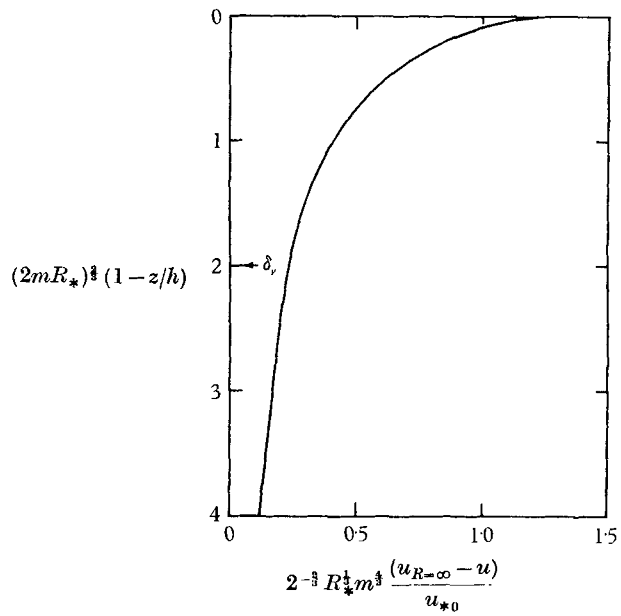

FIGURE 3. Correction to be applied to velocity profiles near the surface to allow for molecular viscosity.

so a little algebra yields

$$\begin{aligned} 2(K_T + \nu) &= \nu + (\nu^2 + 4m^2h^2q^6u_{*0}^2)^{\frac{1}{2}} \\ &= hu_{*0}R_*^{-1}\{1 + (1 + 4m^2R_*^2q^6)^{\frac{1}{2}}\}, \end{aligned} \tag{13}$$

where $R_* = hu_{*0}/\nu$. Hence

$$\begin{aligned} \frac{u_{\max} - u}{u_{*0}} &= \int_0^q \frac{4R_* q'^3\, dq'}{1 + (1 + 4m^2R_*^2 q'^6)^{\frac{1}{2}}} \\ &= 2^{\frac{2}{3}} R_*^{-\frac{1}{3}} m^{-\frac{4}{3}} \int_0^x \frac{x'^3\, dx'}{1 + (1 + x'^6)^{\frac{1}{2}}}, \end{aligned} \tag{14}$$

where $x = (2mR_*)^{\frac{1}{3}} q$.

Since (14) must coincide with the high Reynolds number solution (4) at large values of x, the change induced by molecular viscosity can readily be computed. This is shown in figure 3.

Values of R_* encountered in practice may range from very low values in the laboratory to 10^6 or more in geophysical situations. To take two examples, a channel 10 cm deep with $U = 10\,\text{cm}\,\text{sec}^{-1}$ and a drag coefficient of 0·01 gives $R_* = 10^3$, and an estuary 10 m deep with $U = 200\,\text{cm}\,\text{sec}^{-1}$ and a drag coefficient of 0·0025 gives $R_* = 10^6$. In both these cases it happens (by a coincidence) that with $m = 0{\cdot}8$ the correction to the surface velocity is $0{\cdot}28\,\mathrm{cm\,sec^{-1}}$ and the depth of the viscous sublayer δ_ν (which may be conveniently defined as the depth where molecular and eddy viscosity are equal) is 0·15 cm. The sublayer is thus of slight significance in the channel, and completely negligible in the estuary.

There would be no difficulty in calculating the correction to the apparent longitudinal diffusion coefficient if one were to assume that in the viscous sublayer the transport of pollutant was by molecular diffusion. However, it is known that the viscous but not laminar motions induced by the turbulence outside are important and these complicate the situation. There is also a further practical reason for not computing these corrections; for them to be valid, experiments would have to be conducted over times long compared with that required for a typical fluid particle to enter or leave the sublayer and that would require channels of immense length.

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Appendix

The similarity state at the surface

The theoretical idea behind the dimensional argument used in § 2 is that there is a region near the surface in which the motion is in a state of dynamical similarity with its length scale proportional to $(h-z)$ and its velocity scale proportional to $u_{*0}(1-z/h)^{\frac{1}{2}}$. If this similarity state applies to the whole of the turbulence, it is clear that the fluctuation in horizontal velocity must be proportional to the velocity scale and so fall to zero at the surface, which should therefore move as a rigid sheet. Such strictly laminar motion of the surface does not accord with observation (though this is confined to channels of finite width) nor physical intuition; it thus seems desirable to modify the theory in such a way that the similarity state applies only to certain aspects of the turbulence, which include the eddy viscosity and the shear stress, but exclude the horizontal velocity.

This modification can readily be made if we suppose that the motion is resolved into elements of different sizes (for example, by Fourier analysis) and admit the possibility that even near the surface there may be components in the horizontal motion with a scale comparable with the depth of the channel; since these extend beyond the region of similarity, it is not reasonable to expect the similarity state to include them and its application may be restricted to smaller eddies. The point may become clearer if we consider specifically the one-dimensional spectrum tensor $\Phi_{ij}(\kappa)$ of the turbulent stress tensor $\overline{u_i u_j}$. The similarity law for small eddies near the surface then asserts that

$$\kappa\Phi_{ij} = (1-z/h)\,u_{*0}^2 F_{ij}(\kappa[h-z]), \quad \text{for} \quad \kappa h \gg 1,$$

where F_{ij} is some unknown function of the dimensionless variable $\kappa(h-z)$. Now, if the shear stress $\overline{u_1 u_3}$ depends only on local quantities, there must be at most a negligible contribution to $\int_0^\infty \Phi_{13}\, d\kappa$ from values of κ outside the similarity range and $\int_0^\infty \kappa^{-1} F_{13}\, d\kappa$ must certainly be finite. On the other hand, if there is to be horizontal motion in the surface with a finite amount of energy contained in each part of the spectrum, it is merely necessary that F_{11} should vary as $\kappa^{-1}(h-z)^{-1}$ for small values of $\kappa(h-z)$, since then at $z = h$

$$\kappa\Phi_{11} \propto u_{*0}^2 \kappa^{-1} h^{-1} \quad \text{for} \quad \kappa h \gg 1.$$

Clearly this form for the spectrum does not become small as κ decreases, and indeed would lead to an infinite energy were it not for the cut-off when κ is comparable with h^{-1}. Hence if there are any motions in the surface, they must include some whose scale is comparable with the depth of the channel. Casual observation of natural streams suggests that this is what happens in practice.

The restriction of the similarity state to a limited range of scales is not in any way unusual; a closely analogous situation arises in connexion with the pressure spectrum $\Pi(\kappa)$ at a rigid wall. Close to the wall the stress may be assumed to be approximately constant over a depth δ, and in that case dimensional arguments suggest $\kappa\Pi \propto u_{*0}^4$ and the amplitude of the pressure fluctuations would be infinite if the range of application of the similarity law were not restricted to $z_0^{-1} \gg \kappa \gg \delta^{-1}$.